Kamata, A. (2001) Item Analysis by the Hierarchical Generalized Linear Model. Journal of Educational Measurement, 38, 79-93.

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Motivation

Table 1: Multilevel Structure in Item Responses

Item i (Level 1)	Person j (Level 2)
1	1
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3	1
1	2
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- Repeated measures nested within each person
- Interdependent item responses from the same person

Review of Rasch model

$$P_i(X_{ij} = 1|\theta_j, b_i) = \frac{e^{(\theta_j - b_i)}}{1 + e^{(\theta_j - b_i)}}$$
$$= \frac{1}{1 + e^{-(\theta_j - b_i)}}$$

where
i = 1,...,n
j = 1,...,k
X_{ij} = response of person j to item i (0 or 1)
θ_j = ability for person j
b_i = difficulty for item i

Level-1 model (item level)

Apply logit link function

$$log(\frac{p_{ij}}{1-p_{ij}}) = \beta_{0j} + \beta_{1j}X_{1ij} + ...\beta_{(k-1)j}X_{(k-1)ij}$$

$$= \beta_{0j} + \sum_{q=1}^{k-1} \beta_{qj}X_{qij}$$

where
q = 1,...,k-1 since the dummy variable for the reference item is dropped
X_{ij} is the i^{th} term dummy indicator for person j
β_{0j} is an intercept term of the expected effect of the reference item for person j
β_{qj} is the *difference* of effect for item q from β_{0j}

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Level-2 model (person level)

$$\beta_{0j} = \gamma_{00} + u_{0j}$$
$$\beta_{1j} = \gamma_{10}$$
$$\ldots$$
$$\beta_{(k-1)j} = \gamma_{(k-1)0}$$

where
$u_{0j} \sim N(0, \tau)$

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The intercept β_{0j} is assumed to be varying (random) across persons because items answered by the same person are interdependent.

Equivalence of the HGLM and the Rasch model

Combine level-1 and level-2 models

$$P_{ij} = \frac{1}{1 + exp\{-[u_{0j} - (-\gamma_{i0} - \gamma_{00})]\}}$$

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where
$\theta_j = u_{0j}$
$b_i = -\gamma_{i0} - \gamma_{00}$

Extend to three-level model

Level-2 model (person level)

$$\beta_{0jm} = \gamma_{00m} + u_{0jm}$$
$$\beta_{1jm} = \gamma_{10m}$$
$$\dots$$
$$\beta_{(k-1)jm} = \gamma_{(k-1)0m}$$

where
$u_{0jm} \sim N(r_{00m}, \tau_\gamma)$

u_{0jm} represents the extent to which the ability of person j in school m deviates from the mean ability within school m.
γ_{00m} is an effect of the reference item which varies across schools

Extend to three-level model

Level-3 model (school level)

$$\gamma_{00m} = \pi_{000} + r_{00m}$$
$$\gamma_{10m} = \pi_{100}$$
$$\gamma_{20m} = \pi_{200}$$
$$\dots$$
$$\gamma_{(k-1)0m} = \pi_{(k-1)00}$$

where
$r_{00m} \sim N(0, \tau_\pi)$

Extend to three-level model

$$p_{ijm} = \frac{1}{1 + exp\{-[(r_{00m} + u_{0jm}) - (-\pi_{q00} - \pi_{000})]\}}$$

Compared to the Rasch model

- $-\pi_{q00} - \pi_{000}$ represents item difficulty for item i for i = q (i=1,...,k-1)

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- $r_{00m} + u_{0jm}$ represents the ability for person j in school m

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$$p_{ijm} = \frac{1}{1 + exp\{-[(r_{00m} + u_{0jm}) - (-\pi_{q00} - \pi_{000})]\}}$$

Compared to the Rasch model

- $-\pi_{q00} - \pi_{000}$ represents item difficulty for item i for i = q (i=1,...,k-1)
- $r_{00m} + u_{0jm}$ represents the ability for person j in school m
  - r_{00m} is the random effect associated with school m, interpreted as the average ability of students in school m
  - u_{0jm} is a person-specific ability of person j in school m, interpreted as the extent to which ability of person j deviated from the average ability of students in school m

Application: LSAT dataset

Steps

- Add one additional variable to represent person ID
- Reshape data from wide to long format

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Table 2: Original data (wide)

Q1	Q2	Q3	Q4	Q5
0	0	0	0	0
0	1	0	0	1

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Q1	Q2	Q3	Q4	Q5
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0	1	0	0	1

Table 3: Reshaped data (long)

ID	Item	Response
1	Q1	0
1	Q2	0
1	Q3	0
1	Q4	0
1	Q5	0
2	Q1	0
2	Q2	1
2	Q3	0
2	Q4	0
2	Q5	1

Results

Table 4: Comparison of Item Parameters Estimated by Rasch and MLM

Items	Rasch (mirt)	HGLM (lme4)
Q1	-2.73	2.71
Q2	-1.00	0.99
Q3	-0.24	0.24
Q4	-1.31	1.30
Q5	-2.10	2.08

Notes

- item parameter b estimated via mirt denotes item difficulty
- fixed effect estimated via lme4 denotes item easiness $(b_i = -\gamma_{i0} - \gamma_{00})$